Ranking with Multiple reference Points: Efficient Elicitation and Learning Procedures

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Abstract. We consider the multicriteria ranking problem, and specifically a ranking procedure based on reference points recently proposed in the literature, named Ranking with Multiple reference Points (RMP) [25, 8]. Implementing the RMP method in a real world decision problem requires to elicit the model preference parameters. This can be done indirectly by inferring the parameters from stated preferences, as in [21, 22, 12].

Learning an RMP model from stated preferences proves however to be computationally extremely costly, and can hardly be put in practice using state of the art algorithms. In this paper, we propose a Boolean satisfiability formulation of the inference of an RMP model from a set of pairwise comparisons which is much faster than the existing algorithms.

1 Introduction

The multiple criteria ranking problem consists in computing a preorder on a finite set of alternatives $\mathcal{A}$ when these alternatives are evaluated on multiple criteria. Many ranking methods have been proposed in the literature to tackle this problem. Among ranking methods, the so called *outranking* methods (see e.g., [13, 9]) proceed by comparing alternatives on each criterion, then aggregate these preference relations relative to criteria into a ranking. Actually, with these methods, a ranking is not obtained directly. The preference relations on each criterion are first aggregated into an outranking relation. This is done for each pair of alternatives by considering only the preferences between these alternatives on all criteria, without taking into account the other alternatives. In such a way the independence of irrelevant alternatives (IIA) property of the well known Arrow's impossibility theorem [1] is satisfied. The drawback is that the outranking relation is not transitive in general due to the possible presence of Condorcet cycles [10]. In order to obtain a ranking, a further step, called *exploitation* is applied to the outranking relation. Transitivity is obtained at the cost of loosing the IIA property (which is an unavoidable consequence of Arrow's theorem).

However, outranking methods are well-suited for ranking problems involving qualitative criteria, as they only consider the ordinal aspect of evaluation (as opposed to a cardinal aspect which requires assessing trade-offs between differences of evaluations). A recently proposed outranking based ranking method [25, 8], Ranking with Multiple reference Points (RMP), keeps the specificity of considering ordinal data while fulfilling the IIA property. This statement apparently contradicts Arrow's theorem. Actually, this is not the case, due to the introduction of an additional ingredient, namely the reference points.

Respecting the IIA principle is particularly important when learning ranking models from data (e.g., pairwise comparisons). In particular, when the comparisons involve real alternatives, learning a ranking model from comparisons can lead to a situation where: (*i*) the decision maker states that a is better than b ($a \succ b$), (*ii*) a ranking model $\mathcal{M}$ is computed from a learning set (including $a \succ b$), but (*iii*) when applying $\mathcal{M}$ to the set of alternatives, b is ranked better than a.

To the best of our knowledge, RMP is the only outranking based method which fulfils the IIA property; this ranking method is therefore well suited to be put in practice using learning algorithms that learn an RMP model from a set of pairwise comparisons. In our paper, we propose efficient tools to learn RMP models from data.

The paper is organized as follows. Section 2 introduces the RMP method. In Section 3, we present how to implement the RMP method in practice using algorithms that learn an RMP from pairwise comparisons provided by the Decision Maker (DM). We propose, in Section 4, a standard sequence procedure to elicit an RMP model. Section 5 describes a new efficient algorithm that computes an RMP model from a learning set. This algorithm is based on a Boolean satisfiability formulation. We perform, in Section 6, an empirical analysis of our algorithm to assess its performance as compared to the existing literature. A final Section groups conclusions and further research directions.

2 Ranking with Multiple Points

2.1 Reference points in multicriteria decision aid

Kahneman and Tversky were the first to identify clearly the role of reference points in the formation of preferences in the context of risky [16] and riskless decisions [27]. Reference based preferences have since been studied (see [18, 19]) and multicriteria models using reference points have been proposed to sort alternatives into categories (see e.g. [6, 7]), and to rank alternatives ([25, 8]). In this paper we consider the RMP ranking method [25].

2.2 An introductory example

To introduce how the Ranking with Multiple Points (RMP) method proceeds, we consider a simple illustrative example in which a set of cars are to be ranked from the best to the worst. We consider three cars x, y and z evaluated on the following four criteria: Brakes ([0-10] scale), Road holding ([0-10] scale), Price (€), and Acceleration (seconds to accelerate from 0 to 100km/h). The first two criteria are to be maximized, the last two are to be minimized. The performances of cars are shown in Table 1.

The RMP ranking method makes use of preference parameters to specify the decision maker judgment: (*i*) a set of reference points, and

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	Brakes (Max, [0,10])	Road holding (Max, [0,10])	Price (Min, €)	Acceleration (Min, sec.)
x	9.5	9.5	11.7 K€	29.4 sec.
y	0.5	9.5	11.8 K€	27.9 sec.
z	5	9.5	15.9 K€	26.7 sec.
r^2	8	8	12.0 K€	28.0 sec.
r^1	2	4	18.0 K€	31.0 sec.

Table 1. Illustrative example

(*ii*) an importance relation on criteria coalitions (in this example, all criteria are assumed equally important, and it is sufficient to count criteria in coalitions to compare them).

In our example, we use two reference points (which are vectors of evaluations), r^1 and r^2, such that r^2_j is better than r^1_j on each criterion j. These two reference points define three segments of performances on each criterion:

- better than r^2 (which can be interpreted as "*good*"),
- between r^1 and r^2 (which can be interpreted as "*intermediate or fair*"); and
- worse than r^1 (which can be interpreted as "*insufficient*").

The values of these points r^1 and r^2 on criteria are provided in Table 1. For instance, on the criterion "Brakes", any alternative evaluated 8 or above will be considered "good" (e.g., alternative x) and any alternative evaluated lower than 2 will be considered "insufficient" (e.g., alternative y). In other terms, the reference points allow to identify an ordered encoding for each criterion defined by 3 ordered intervals of performances (A, B and C) as illustrated in Figure 1, such that:

A performances above r^2 on each criterion are denoted as A (which can be interpreted as "*good*").

B performances between r^1 and r^2 on each criterion are denoted as B (which can be interpreted as "*intermediate or fair*").

C performances below r^1 on each criterion are denoted as C (which can be interpreted as "*insufficient*")

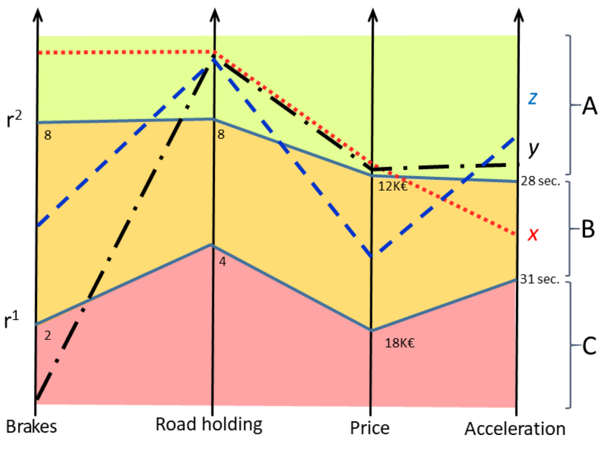

Figure 1. Graphical interpretation of Table 1

The RMP method ranks alternatives based on these ordered intervals of performances. Table 2 shows the results of the encoding for the 3 alternatives considered in our example. For instance, z is encoded B on criterion "Brakes" because z is worse than r^2 but better than r^1.

	Brakes	Road Holding	Price	Acceleration
x	A	A	A	B
y	C	A	A	A
z	B	A	B	A

Table 2. Results of the encoding procedure for the illustrative example

To compute a ranking, alternatives are not compared one to each other but compared to the reference points. Alternatives are compared to the first reference point r^1. Considering two alternatives a and b, a is preferred to b, noted $a \succ b$, if the coalition of criteria for which alternative a is evaluated A or B (i.e. better than r^1) is more important than the coalition of criteria for which alternative b is evaluated A or B (i.e., better than r^1). In this example, criteria are assumed equally important, so we just count the number of criteria. If a and b cannot be distinguished with respect to their comparison to r^1, then a and b are compared to r^2. If the number of criteria for which alternative a is evaluated A (i.e. better than r^2) is greater than the number of criteria for which alternative b is evaluated A (i.e., better than r^2), then a is preferred to b, otherwise a is indifferent to b. In our example, we thus have the following:

- Alternative x is better than y because x has evaluation A or B for all criteria, while y has evaluation A or B for only three criteria (x compares better to r^1 than y does).
- Alternative x is better than z because x and z are both evaluated A or B on all criteria (they compare equally to r^1), but x is evaluated A on three criteria while z is evaluated A only on two criteria (x compares better to r^2 than z does).
- Alternative z is better than y because z has evaluation A or B on all criteria while y has evaluation A or B on three criteria only (z compares better to r^1 than y does).

2.3 The RMP ranking method

We consider $\mathcal{A}$, a set of alternatives evaluated on n criteria. Let us denote $\mathcal{N} = \{1, 2, \ldots, i, \ldots, n\}$ the set of criteria indices, and a_i denotes the evaluation of alternative $a \in \mathcal{A}$ on criterion i (in what follows we will consider, without loss of generality, that preferences increase with the evaluation on each criterion, i.e., the greater the better). The RMP method is a method for ranking a finite set of alternatives evaluated on several criteria [25].

To rank alternatives, RMP compares alternatives to reference points, and then aggregates these comparisons into a final ranking. A dominance structure can be assumed on the set of reference points without loss of generality (for any RMP model using a set of reference points without any dominance structure, there exist an equivalent RMP model using a set of reference points with a dominance structure). RMP makes use of two types of preference parameters:

- $\mathcal{R} = \{r^1, r^2, \ldots, r^h, \ldots, r^m\}$, with $r^h = \{r^h_1, ..., r^h_i, ..., r^h_n\}$, where r^h_i denotes the evaluation of r^h on criterion i;
- an importance relation on criteria coalitions, $\trianglerighteq \subseteq \mathcal{P}(\mathcal{N})$, where $\triangleright$ and $\equiv$ represent the asymmetric and symmetric part of $\trianglerighteq$.

RMP proceeds through the following three steps:

1. compute $c(a, r^h) = \{i \in \mathcal{N} : a_i \geq r_i^h\}$, $a \in \mathcal{A}$, $h = 1, \ldots m$, the set of criteria on which alternative a is at least as good as the reference point r^h.
2. compare alternatives one to each other to define k preference relations $\succsim_{r^h}$ relative to each reference point such that $a \succsim_{r^h} b$ iff $c(a, r^h) \trianglerighteq c(b, r^h)$. In other words, $a \succsim_{r^h} b$ holds when a compares better to r^h than b does. We denote $\succ_{r^h}$ ($\sim_{r^h}$, respectively) the asymmetric part of the relation $\succsim_{r^h}$ (the symmetric part of $\succsim_{r^h}$, respectively).
3. to rank two alternatives $a, b \in \mathcal{A}$, consider sequentially the relations $\succsim_{r^1}, \succsim_{r^2}, \ldots, \succsim_{r^k}$; a is preferred to b if $a \succ_{r^1} b$, or if $a \sim_{r^1} b$ and $a \succ_{r^2} b$, or ... Hence, a and b are indifferent iff $a \sim_{r^h} b$, for all $h = 1 \ldots m$.

Rolland [25] proved that by proceeding in such a way, the computed preference relations on alternatives are guaranteed to be transitive. As mentioned earlier, a dominance structure on the set of reference points can be assumed without loss of generality.

3 Implementing the RMP ranking method

To implement the RMP method in a decision aiding study, an interaction with the DM is required, so as to integrate her preferences, hence set the values of the preference parameters involved in the RMP method. A basic approach called *direct elicitation* consists in interacting with the DM directly on the values of the preference parameters. However, such an approach is not recommended as the DM usually has no clear understanding of the semantics attached to the preference parameters. Moreover, it imposes a strong cognitive burden on the DM. Therefore, the literature frequently proposes an *indirect elicitation*, in which the DM expresses holistic preferences (i.e., pairwise comparisons of alternatives) from which the values of the preference parameters are inferred (see e.g. [5, 15, 24]).

Recent literature (see [21, 28]) proposed indirect elicitation procedures for the S-RMP method (a particular case of RMP in which the criteria importance relation is additively representable). The Decision Maker provides pairwise comparisons of alternatives from which the S-RMP preference parameters (weights, reference points, and the lexicographic order on reference points) are inferred. Two algorithms were proposed:

- **MIP-based algorithm.** [28, 21] formulate the elicitation of a S-RMP model as a mixed linear optimization problem. In this optimization program, the variables are the parameters of the S-RMP method, and additional technical variables which enable to formulate the objective function and the constraints in a linear form. The aim is to minimize the Kemeny distance (see [17]) between the partial ranking provided by the Decision Maker (i.e. the comparisons) and the S-RMP ranking. The resolution of this optimization program provides a guarantee that the elicited S-RMP model best matches the pairwise comparisons in terms of the Kemeny distance between the comparisons provided by the DM and the S-RMP ranking.

- **Metaheuristic algorithm.** Another algorithm to indirectly elicit an S-RMP model, from pairwise comparisons, was proposed by [22, 21]. Unlike the MIP version, this metaheuristic does not guarantee that the inferred model is the one which minimizes the Kemeny distance to DM's statements. Indeed, the perspective is to obtain an S-RMP model which fits the Decision Maker's comparisons "well" within a "reasonable" computing time. This metaheuristic is based on an evolutionary algorithm in which a population of S-RMP models is iteratively evolved.

The above mentioned algorithms suffer however from limitations:

- both algorithms only consider an additive representation of criteria importance relation, which can be restrictive when interaction between criteria occur;
- the MIP based approach is not able to deal with datasets whose size correspond to real world decision problems (e.g. 10 criteria, 2 reference points and 50 comparisons);
- the heuristic approach is fast but is not always able to restore an S-RMP model compatible with a set of comparisons, whenever it exists.

To circumvent these limitations, two paths are possible:

- elicit an RMP model using a *model-based elicitation strategy* analogous to the one described in [3] for the NonCompensatory Sorting model [6, 7]. This approach permits to elicit the RMP parameters by asking the decision maker to make comparisons, and aims at building the shortest questionnaire. We propose in Section 4 such a procedure for RMP with one single preference point.
- Design an algorithm similar to the MIP approach that can handle real-world size datasets, as done for the NonCompensatory Sorting model [6, 7] to overcome computational issues of [20] using a Boolean satisfaction (SAT) formulation, see [2]. In this perspective, we propose, in Section 5, a SAT formulation which is computationally efficient.

4 A procedure to elicit an RMP model

In this section, we restrict ourselves to RMP with a single reference point, and we propose an elicitation procedure in which the DM answers a sequence of questions that will lead to a complete knowledge of the RMP parameters (the importance relation on coalitions, and the reference point).

This procedure is structured in two consecutive phases: in the first phase, the answers of the DM leads to define the $\trianglerighteq$ importance relation on criteria coalitions, the reference point being unknown, while the second phase aims at specifying the reference point. The possibility to identify, in the first phase, the $\trianglerighteq$ relation without knowledge on the reference point is based on the following remark. Consider the alternative x_A, with $A \subseteq \mathcal{N}$, having the best possible evaluation on criterion $i \in A$, and the worst possible evaluation on criterion $j \in \mathcal{N} \backslash A$. if $x_A \succ x_B$, then $A \trianglerighteq B$ and not $[B \trianglerighteq A]$ hold whatever the reference point. Hence, it is possible to determine $\trianglerighteq$ in the absence of knowledge on the reference point (note, however, that this is possible only with RMP models involving a single reference point).

The first phase of the algorithm aims at eliciting the $\trianglerighteq$ relation. Let us first recall that the relation $\trianglerighteq$ defined on $\mathcal{P}(\mathcal{N})$ is transitive and compatible with inclusion, i.e, for any pair of criteria coalitions $A, B \subseteq \mathcal{N}$, $B \subset A \Rightarrow A \trianglerighteq B$. Consider the "*minimal*" relation $\trianglerighteq^0$ containing pairs of coalitions corresponding to inclusion situations. Consider two coalitions that are not in $\trianglerighteq^0$. A positive answer to the question "*is x_A preferred to x_B*" will enrich $\trianglerighteq^0$ with the statement $A \trianglerighteq B$, and all transitive consequences ($A' \trianglerighteq B'$, for all A', B' such that $A \subseteq A'$ and $B' \subseteq B$).

Hence, the answer to the question "*Is x_A preferred to x_B?*" will enrich relation $\trianglerighteq$, and we can proceed so until $\trianglerighteq$ corresponds to a complete pre-order. In other words, $\trianglerighteq^0$ should be completed to reach a complete and transitive relation on the subsets of $\mathcal{N}$, in which case the importance relation on coalitions is fully known. Obviously, the order by which questions are posed should be defined so as to minimize the total number of questions. This issue is not discussed in this paper.

The second phase of the algorithm aims at eliciting the reference point r given the elicited relation $\trianglerighteq$. In order to elicit r_i the evaluation of the reference point on criterion i, consider two coalitions A and B, such that $i \notin B$, $A \trianglerighteq B$ and not $[A \trianglerighteq B \cup \{i\}]$. By construction, we have $x_A \succsim x_B$, but not $x_A \succsim x_{B\cup\{i\}}$. Consider now the alternative $x_B^{k_i}$ having the same evaluations as x_B except on criterion i on which its evaluation is k_i. If $x_A \succsim x_B^{k_i}$ holds, then it means that $k_i < r_i$. From the preceding implication, we can design a dichotomous search to elicit r_i from questions of the type "*Is x_A preferred to $x_B^{k_i}$?*". Proceeding in this way for each criterion leads to elicit r. Note that r_i can also be elicited analogously considering two coalitions A and B, such that $i \notin A$, not $A \trianglerighteq B$ and $A \cup \{i\} \trianglerighteq B$.

5 Learning an RMP model from pairwise comparisons: a SAT formulation model

In this section, we propose a new procedure to check whether a set of pairwise comparisons can be represented by an RMP model with k reference points using a Boolean satisfiability (SAT) formulation.

5.1 Boolean satisfiability (SAT)

A Boolean satisfaction problem consists of a set of Boolean variables V and a logical proposition about these variables $f : \{0,1\}^V \to \{0,1\}$. A solution $v^\star$ is an assignment of the variables mapped to 1 by the proposition: $f(v^\star) = 1$. A binary satisfaction problem for which there exists at least one solution is *satisfiable*, else it is *unsatisfiable*. Without loss of generality, the proposition f can be assumed to be written in conjunctive normal form: $f = \bigwedge_{c\in\mathcal{C}} c$, where each *clause* $c \in \mathcal{C}$ is itself a disjunction in the variables or their negation $\forall c \in \mathcal{C}, \exists c^+, c^- \in \mathcal{P}(V) : c = \bigvee_{v\in c^+} v \vee \bigvee_{v\in c^-} \neg v$, so that a solution satisfies at least one condition (either positive or negative) of every clause.

The models presented hereafter make extensive use of clauses where there is only one non-negated variable (a subset of *Horn clauses*): $a \vee \neg b_1 \vee \cdots \vee \neg b_n$, which represent the logical implication $(b_1 \wedge \cdots \wedge b_n) \Rightarrow a$. It is known since Cook's theorem [11] that the Boolean satisfiability problem is NP-complete. Consequently, unless $P = NP$, we should not expect to solve generic SAT instances quicker than exponential time in the worst case. Nevertheless, efficient and scalable algorithms for SAT have been – and still are – developed, and are sometimes able to handle problem instances involving tens of thousands of variables and millions of clauses in a few seconds (see e.g. [23, 4]).

5.2 A SAT encoding of given comparisons in RMP

We consider a set $\mathcal{BC} = \bigcup_{j\in\mathcal{J}}\{p^j \succ n^j\}$ of binary comparisons provided by the DM, (p for "positive", n for "negative"). Below, we will use the following indices:

- $h \in \mathcal{H}$ is an index for reference points ordered by importance (i.e. to compare alternatives, we consider r^1, then r^2 if needed, etc.);
- $i \in \mathcal{N}$ is the index for criteria;
- $j \in \mathcal{J}$ is the index for comparisons in the learning set, composed of pairs $p^j \succ n^j$ (p for "positive", n for "negative"), where $p^j = (p_1^j, p_2^j, \ldots, p_n^j)$ and $n^j = (n_1^j, n_2^j, \ldots, n_n^j)$ are evaluation vectors;
- $k \in \mathbb{X}_i$ denotes values taken on criterion $i \in N$ (i.e. the evaluation scale on criterion i is $\mathbb{X}_i = \bigcup_{j\in\mathcal{J}}\{p_i^j, n_i^j\}$).

We introduce the following variables:

- $x_{i,h,k}$ take value 1 iff the value k is above the reference point r^h on criterion i ($k \geq r_i^h$);
- $y_{A,B}$ take value 1 iff the criteria coalition A is more important than coalition B;
- $z_{j,h}$ take value 1 iff criteria for which alternative p^j is above reference point r^h are at least as important as those for which alternative n^j is above r^h $\big(c(p^j, r^h) \trianglerighteq c(n^j, r^h)\big)$;
- $z'_{j,h}$ take value 1 iff criteria for which alternative n^j is above reference point r^h are at least as important as those for which alternative p^j is above r^h $\big(c(n^j, r^h) \trianglerighteq c(p^j, r^h)\big)$;
- $d_{h,h'}$ take value 1 iff the reference point r^h dominates reference point $r^{h'}$ ($r_i^h \geq r_i^{h'}, \forall i \in \mathcal{N}$);
- $s_{j,h}$ take value 1 iff alternative p^j is indifferent to alternative n^j with respect to all reference points $r^{h'}$, with $h' < h$, and p^j compares to reference point r^h at least as well as n^j does;

Definition 1 (SAT encoding for RMP)**.** *Consider $\mathcal{BC} = \{(p^j, n^j), j \in \mathcal{J}\}$ a set of binary comparisons ($p^j \succ n^j$). We define the Boolean function $\phi_{\mathcal{BC}}^{SAT}$ as the conjunction of clauses:*

- *For all criteria $i \in \mathcal{N}$, for all reference point r^h, for all pairs of values $k, k' \in \mathbb{X}_i$ such that $k < k'$:*

$$x_{i,h,k} \vee \neg x_{i,h,k'} \tag{1}$$

- *For all pairs of reference points $r^h, r^{h'}$ such that $h < h'$:*

$$d_{h,h'} \vee d_{h',h} \tag{2a}$$

- *For all criteria $i \in \mathcal{N}$, for value $k \in \mathbb{X}_i$, for all pairs of reference points $r^h, r^{h'}$ such that $h \neq h'$:*

$$x_{i,h',k} \vee \neg x_{i,h,k} \vee \neg d_{h,h'} \tag{2b}$$

- *For all pairs of coalitions $A, B \subseteq \mathcal{N}$:*

$$y_{A,B} \vee y_{B,A} \tag{3a}$$

- *For all pairs of coalitions $A, B \subseteq \mathcal{N}$ such that $A \subset B$:*

$$y_{B,A} \tag{3b}$$

- *For all pairs of coalitions $A, B, C \subseteq \mathcal{N}$:*

$$\neg y_{A,B} \vee \neg y_{B,C} \vee y_{A,C} \tag{3c}$$

- *For all pairs of coalitions $A, B \subseteq \mathcal{N}$, for all comparisons $j \in \mathcal{J}$, for all reference point $r^h, h \in \mathcal{H}$:*

$$\bigvee_{i\notin A} x_{i,h,p_i^j} \vee \bigvee_{i\in B} \neg x_{i,h,n_i^j} \vee y_{A,B} \vee \neg z_{j,h} \tag{4a}$$

- *For all pairs of coalitions $A, B \subseteq \mathcal{N}$, for all comparisons $j \in \mathcal{J}$, for all reference point $r^h, h \in \mathcal{H}$:*

$$\bigvee_{i\notin A} x_{i,h,n_i^j} \vee \bigvee_{i\in B} \neg x_{i,h,p_i^j} \vee y_{A,B} \vee \neg z'_{j,h} \tag{4b}$$

- *For all pairs of coalitions $A, B \subseteq \mathcal{N}$, for all comparisons $j \in \mathcal{J}$, for all reference point $r^h, h \in \mathcal{H}$:*

$$\bigvee_{i\in A} \neg x_{i,h,p_i^j} \vee \bigvee_{i\notin A} x_{i,h,p_i^j} \vee \bigvee_{i\in B} \neg x_{i,h,n_i^j} \vee \bigvee_{i\notin B} x_{i,h,n_i^j} \vee \neg y_{B,A} \vee z'_{j,h} \tag{4c}$$

- *For each comparison $j \in \mathcal{J}$:*

$$\bigvee_{h \in H} s_{j,h} \tag{4d}$$

- *For each comparison $j \in \mathcal{J}$, for all pairs of reference points $r^h, r^{h'}; h, h' \in \mathcal{H}$ such that $h < h'$:*

$$z_{j,h} \vee \neg s_{j,h'} \tag{5a}$$

- *For each comparison $j \in \mathcal{J}$, for all pairs of reference points $r^h, r^{h'}; h, h' \in \mathcal{H}$ such that $h < h'$:*

$$z'_{j,h} \vee \neg s_{j,h'} \tag{5b}$$

- *For all reference points $r^h, h \in \mathcal{H}$:*

$$\neg z'_{j,h} \vee \neg s_{j,h} \tag{5c}$$

In Definition 1, clauses (1) impose that evaluation scale is monotone with respect to reference points on each criterion $i \in \mathcal{N}$. It states that if evaluation k is above r^h on criterion i, then any evaluation $k' > k$ is also above r^h on criterion i (we assume without loss of generality that all criteria are to be maximized).

Clauses (2a-2b) impose a dominance structure on reference points. (2a) check that, for any pair of reference points, either r^h dominates $r^{h'}$ or the reverse. Clauses (2b) relate variables $x_{i,h,k}$ to variables $d_{h,h'}$ stating that if, on criterion i, evaluation k is above reference point r^h, but not above reference point $r^{h'}$, then $r^{h'}$ does not dominate r^h.

Clauses (3a-3c) guarantee that the importance relation $\trianglerighteq$ on criteria coalitions is consistently defined. Clauses (3a) ensure relation $\trianglerighteq$ to be complete, clauses (3b) ensure that $\trianglerighteq$ is compatible with inclusion, and clauses (3b) impose transitivity.

Clauses (4a-4d) guarantee that the pairs p_j, n_j compare such that $p_j \succ n_j$. Clauses (5a-5c) guarantee that, for any comparison $j \in \mathcal{J}$, when p_j and n_j are separated by reference point$r^{h'}$, p_j and n_j are indifferent with respect to all reference points r^h such that $h < h'$.

6 Numerical investigation of the SAT formulation

In this section, we study the performance of the formulation proposed in section 5.2, both intrinsic and comparative with respect to state-of-the-art techniques. We use a state-of-the-art SAT solver, in order to solve instances of the problem of learning an RMP model, given a set of pairwise comparisons. We begin by describing our experimental protocol, with some implementation details. Then, we provide the results of the experimental study concerning the computation time of our algorithm, and particularly the influence of the size of the learning set, and the number of criteria, as well as elements of comparison between existing approaches.

6.1 Experimental design

The algorithm we test takes as input a set of pairwise comparisons in which alternatives compared are described by a performance tuple on a set of criteria $\mathcal{N}$.

The performance is measured in practice, by solving actual instances of the problem and reporting the computation time required. This experimental study is run on an ordinary laptop running under linux, equipped with an i7-6600U CPU at i2.6 GHz and 20 GB of RAM.

Dataset generation.

In the scope of this paper, we only consider to use a carefully crafted, random dataset as an input. On the one hand, the algorithm we describe is not yet equipped with the capability to deal with noisy inputs, so we do not consider feeding it with actual preference data, such as the one found in preference learning benchmarks [14]. On the other hand, using totally random, unstructured, inputs makes no sense in the context of algorithmic decision. Hence, we use a decision model to generate it, and, in particular, a model compatible with the RMP model. Precisely, we use a S-RMP model for generating the learning set, a model that particularizes RMP by postulating the set of importance relation on criteria coalitions possess an additive structure (i.e., there is a set of weights $w_i, i \in \mathcal{N}$, with $w_i \geq 0, \forall i$ and $\sum_i w_i = 1$, such that $A \trianglerighteq B$ iff $\sum_{i \in A} w_i \geq \sum_{i \in B} w_i$). This choice ensures our SAT formulation should succeed in finding the parameters of a model compatible with all the pairwise comparisons in the input.

When generating a dataset, we consider the number of criteria $|\mathcal{N}|$, the number of comparisons $|\mathcal{J}|$, and the number of reference points m as experimental parameters.

We consider all criteria take continuous values in the interval $[0, 1]$. We generate a set of m reference points $\langle r \rangle$ by uniformly sampling m numbers in the interval $[0, 1]$ and sorting them in ascending order, for all criteria; we then randomly re-order the reference points. We generate criteria weights $\langle w \rangle$ by sampling $|\mathcal{N}| - 1$ numbers in the interval $[0, 1]$, sorting them, and using them as the cumulative sum of weights.

Finally, we sample uniformly pairs of tuples in $[0, 1]^{\mathcal{N}}$, defining the performance of two alternatives[3], compare these two alternatives with $\mathcal{M}^0$:=S-RMP$_{m,\langle r \rangle,\langle w \rangle}$ and consequently determine which one is p^j and n^j, $j \in \mathcal{J}$.

Solving the SAT problem.

For a given number of criteria $|\mathcal{N}|$, a given number of reference points m, we check if a given set $\mathcal{BC}$ of binary comparisons can be represented by the RMP model, by solving the corresponding SAT formulation presented in §5.2, using the SAT solver CryptoMiniSAT 5.0.1 [26], winner of the incremental track at SAT Competition 2016 (http://baldur.iti.kit.edu/sat-competition-2016/). If the solver finds a solution, then it is converted into parameters $(\langle r^{\mathrm{SAT}} \rangle, \trianglerighteq^{\mathrm{SAT}})$ for an RMP model. The model $\mathcal{M}^{\mathrm{SAT}}$ =RMP$_{\langle r^{\mathrm{SAT}} \rangle, \trianglerighteq^{\mathrm{SAT}}}$ yielded by the program is then validated against the input. As the ground truth $\mathcal{M}^0$ used to generate the binary comparisons is an S-RMP model (and therefore an RMP model), we expect the solver to always find a solution, and we expect the RMP model returned by the program to always succeed at restoring the provided comparisons.

Ability of the inferred models to restore the original one.

In order to appreciate how "close" a computed model $\mathcal{M}^{SAT}$ is to the ground truth $\mathcal{M}^0$ from which the comparisons were generated, we proceed as follows: we sample a set of 10000 pairs of tuples in $X = [0, 1]^{\mathcal{N}}$ and compute the comparisons of these pairs according to the original and computed RMP models ($\mathcal{M}^0$ and $\mathcal{M}^{SAT}$). On this basis, we compute $err - rate$ the proportion of "errors", *i.e.* pairs which do not compare in the same way by both models.

[3] Only pairs of tuples that are not in the dominance relation are kept.

6.2 Performance of the SAT formulation

We run the above described experimental protocol varying the various values of the parameters: (*i*) the number of criteria $|\mathcal{N}|$ is chosen among $\{3, 4, 5\}$, (*ii*) the number of comparisons $|\mathcal{BC}|$ is chosen among $\{100, 200, ..., 1000\}$, and (*iii*) the number of reference points m is chosen among $\{1, 2, 3\}$. For each value of the triplet of parameters, we sample 10 S-RMP models $\mathcal{M}^0$, and record the computation time (t) needed to provide a model $\mathcal{M}^{\text{SAT}}$.

6.2.1 *Results regarding computation time.*

Figures 2 and 3 show the average computing time required to infer the parameters of one RMP model when the number of examples, criteria and reference points vary. We see in Fig. 2 that the computing time seems to grow exponentially as a function of the number of criteria. Indeed, when the reference set contains 500 alternatives, the average computing times for 2 reference points and 3, 4 and 5 criteria are respectively equal to about 1.5 seconds, 15 seconds and 75 seconds. It is no surprise since the number of constraints in the SAT formulation evolves as well exponentially as the number of criteria grows. When we vary the number of reference points (Figure 3), we observe that the same phenomenon occurs. Indeed, for 500 pairwise comparisons in the learning set, the average computing time is about 20 seconds when the model has one reference point, it grows up to ± 60 seconds for two reference points and up to ± 250 seconds for 3 reference points. For an RMP model with a fixed number of criteria and reference points, we see both in Figures 2 and 3 that the computing time evolves linearly when the number of pairwise assignment increases. Again, this is no surprise since the number of constraints involved also tends to increase linearly.

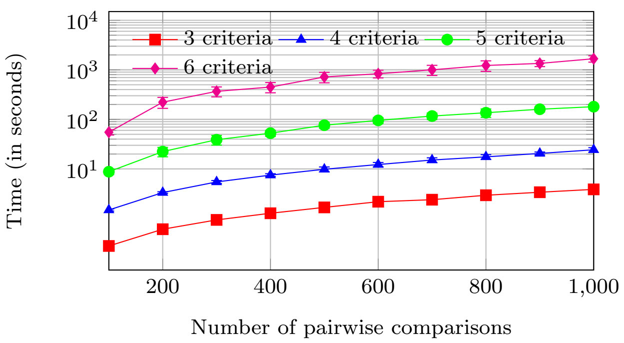

Figure 2. Computing time as a function of the number of pairwise comparisons for models involving 2 reference points and 3 to 6 criteria. Bars represent standard deviation.

6.2.2 *Results on the ability of the inferred model to restore the original one.*

To assess the ability of the SAT formulation to restore a model that is the closest to the original one, we sample a set of 10000 pairwise comparisons and we compute their relation of preference both with the original model ($\mathcal{M}^0$) and the one learned with the SAT solver ($\mathcal{M}^{\text{SAT}}$). Then we compute the proportion of binary comparisons that have the same preference relation with $\mathcal{M}^0$ and $\mathcal{M}^{\text{SAT}}$.

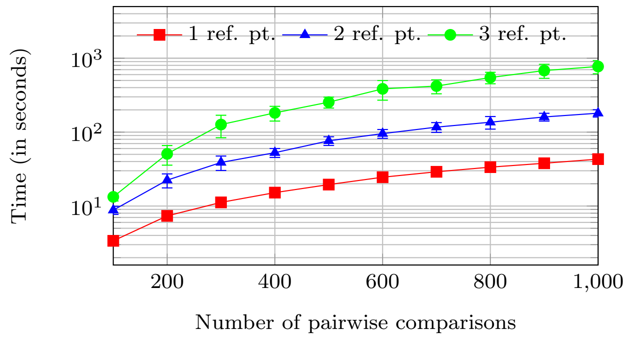

Figure 3. Computing time as a function of the number of pairwise comparisons for models with 5 criteria, and 1 to 3 reference points. Bars represent standard deviation.

In Figures 4 and 5, we observe that the average number of pairs of alternatives from the test set that have the same preference relation both with $\mathcal{M}^0$ and $\mathcal{M}^{\text{SAT}}$ increases as a function of the number of pairs in the learning set. When the number of criteria increases, the number of pairs required to restore the original model $\mathcal{M}^0$ increases. Figure 4 shows that with 100 alternatives, it is possible to restore on average more than 90 percent of the relations. With 6 criteria and a learning set of 100 pairs, less than 80 percent of the pairwise relations are restored. The same observation holds when the number of reference points increases (see Figure 5).

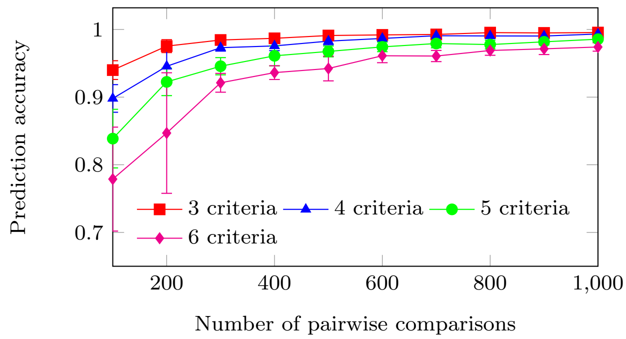

Figure 4. Average prediction accuracy as a function of the number of pairwise comparisons for models involving 2 reference points, and 3 to 6 criteria. Test set of 10000 pairwise comparisons. Bars represent standard deviation.

6.3 Discussion

Experimental results have shown that the algorithm was efficient for inferring an RMP model from large sets of binary comparisons. Indeed, the formulation is able to restore an RMP model composed of 3 reference points and 5 criteria from 500 binary comparisons in more or less 250 seconds. Furthermore, the algorithm performs well in generalisation. With barely 100 alternatives, the SAT formulation can learn an RMP model that predicts more than 70% of the binary relations obtained with a S-RMP model.

It should be highlighted that such a performance proves this formulation to be superior to existing algorithms. Indeed, MIP based

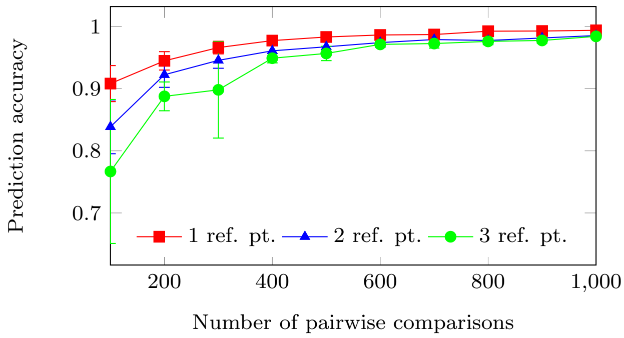

Figure 5. Average prediction accuracy as a function of the number of pairwise comparisons for models involving 5 criteria, and 1 to 3 reference points. Test set of 10000 pairwise comparisons. Bars represent standard deviation.

algorithms [28] are only able to handle a few dozens of pairwise comparisons which is insufficient to infer an RMP model with good generalization ability. Heuristic approaches [22, 21] can handle larger datasets, but are not able to systematically restore an RMP compatible input. A drawback of our approach is however its inability to easily handle noisy input.

7 Conclusion

In this paper, we describe a SAT formulation in order to learn an RMP model from a set of binary comparisons. Experimental results show that the algorithm is efficient enough to deal with large datasets and performs well in generalization. This formulation can be solved more efficiently than the MIP [28] and is more acurate than the heuristic approach [22, 21]. Our proposal is a step forward toward the possibility of eliciting an RMP model in an interactive process with the DM.

We see several research that should be pursued. The formulation presented in this paper can only deal with datasets that do not contain errors. A path to explore consists in finding a formulation that is able to handle errors, for instance by using a MAXSAT formulation. In this paper, the experiments have been done on artificial datasets. Another path to explore consists in using it with real datasets like in [12].

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