UTA-poly and UTADIS-poly: using polynomial marginal utility functions in UTA and UTADIS

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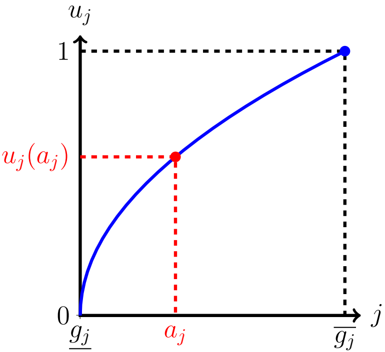
Additive utility function model

- A **marginal utility function** u_j is associated to each criterion j
- Marginal utility functions are **monotonic**
- Marginal utility functions are **normalized** between 0 and 1, s.t. $u_j(\underline{g_j}) = 0$ and $u_j(\overline{g_j}) = 1$
- A **weight** w_j is associated to each criterion j, s.t. $\sum_j w_j = 1$



- Utility of an alternative *a* :

$$U(a) = \sum_{j=1}^{n} w_j \cdot u_j(a_j)$$

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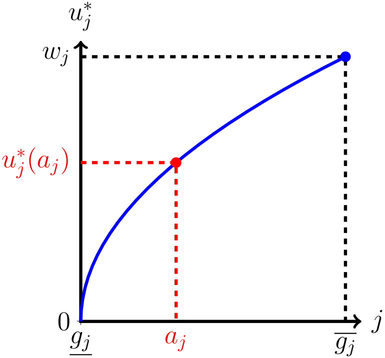


- We also have : $u_j^*(a) = w_j \cdot u_j(a_j)$ and $u_j^*(\overline{g_j}) = w_j$
- Utility of an alternative a :

$$U(a) = \sum_{j=1}^{n} w_j \cdot u_j(a_j) = \sum_{j=1}^{n} u_j^*(a_j)$$

UTA : Presentation

- Not easy to elicit directly the marginal utility functions
- **Disaggregation** procedure proposed by [Jacquet-Lagrèze and Siskos, 1982]
- Utility functions are computed on basis of a ranking given in input
- Linear programming

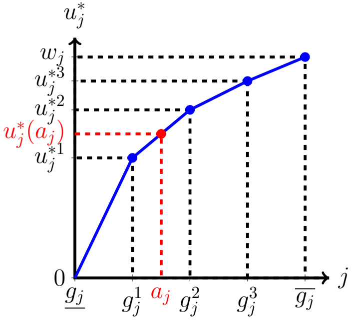
UTA - Constraints I

- Two type of information (pairwise comparison) :
  1. a is preferred to b, i.e. $U(a) > U(b) \Rightarrow (a, b) \in \mathcal{P}$
  2. a is indifferent to b, i.e. $U(a) = U(b) \Rightarrow (a, b) \in \mathcal{I}$
- A potential error is introduced for each alternative utility $U(a)$, s.t. $U'(a) = U(a) + \sigma^+(a) - \sigma^-(a)$
- Constraints of the linear program :

$$\begin{cases} U(a) - U(b) + \sigma^+(a) - \sigma^-(a) & & & \\ \quad -\sigma^+(b) + \sigma^-(b) & > & 0 & \forall (a,b) \in \mathcal{P}, \\ U(a) - U(b) + \sigma^+(a) - \sigma^-(a) & & & \\ \quad -\sigma^+(b) + \sigma^-(b) & = & 0 & \forall (a,b) \in \mathcal{I}, \\ \sum_{j=1}^{n} u_j^*(\overline{g_j}) & = & 1, & \\ \sum_{j=1}^{n} u_j^*(\underline{g_j}) & = & 0, & \\ \sigma^+(a), \sigma^-(a) & \geq & 0 & \forall a \in A^*, \\ u_j^* & \text{monotonic} & & \forall j \in N. \end{cases}$$

UTA - Constraints II

- Monotonicity is ensured by using **piecewise linear functions** for the marginal utility functions



- Domain of the criterion split in k equal parts
- Position of the g_j^l, for $l = 0, \ldots, k$ fixed a priori (equidistant)

- Marginal utility value of an alternative a :

$$u_j^*(a) = u_j^{*L-1} + \left(\frac{a_j - g_j^{L-1}}{g_j^L - g_j^{L-1}}\right)\left(u_j^{*L} - u_j^{*L-1}\right)$$

with g_j^L the first breakpoint s.t. $a_j \leq g_j^L$

UTADIS

- Sorting problems
- Comparison of alternatives to thresholds delimiting the categories
- Disaggregation though linear programming
- Similar approach as for UTA : utility functions are modelled through piecewise linear functions

Motivations for UTA-poly and UTADIS-poly I

Drawbacks of UTA methods

- Marginal utility functions are not natural close to the breakpoints of the piecewise linear functions
- Breakpoints at pre-defined position : limit the flexibility of the model

Existing works

- Bugera, V., Konno, H., and Uryasev, S. (2002). Credit cards scoring with quadratic utility functions.
  Journal of Multi-Criteria Decision Analysis, 11(4-5):197–211
- Słowínski, R., Greco, S., and Mousseau, V. (2005). Multi-criteria ranking of a finite set of alternatives using ordinal regression and additive utility functions - a new UTA-GMS method.
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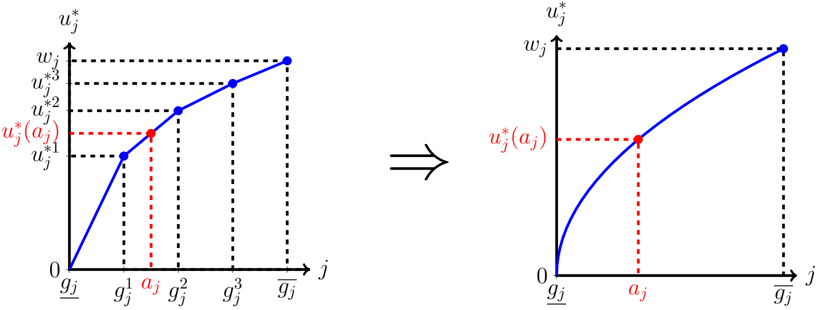
Motivations for UTA-poly and UTADIS-poly II

UTA-poly and UTADIS-poly

- We propose to replace piecewise linear functions by polynomial ones by using semi-definite programming (SDP)
- Degree of the polynomial chosen a priori



$$u_j^*(a) = u_j^*(g_j^1) + \left(\frac{a_j - g_j^1}{g_j^2 - g_j^1}\right)\left(u_j^{*2} - u_j^{*1}\right)$$

$$u_j^*(a) = p_0 + p_1 \cdot a_j + p_2 \cdot a_j^2 + \ldots + p_D \cdot a_j^D$$

Semi-Definite Programming (SDP) I

Theorem

A polynomial $F(z)$, with $z \in \mathbb{R}^n$ is nonnegative if it is possible to decompose it as a sum of squares (SOS) :

$$F(z) = \sum_s f_s^2(z) \qquad \text{with } f_s(z) \in \mathbb{R}^n.$$

Not every non-negative polynomial is a Sum Of Squares (SOS), but :

Theorem

(Hilbert) A non-negative polynomial in one variable is always a SOS.

Semi-Definite Programming (SDP) II

- Consider the following polynomial of degree D :

$$p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_D x^D = \sum_{i=0}^{D} p_i x^i.$$

$$\boxed{p(x) \text{ non-negative} \iff \text{it can be decomposed as a SOS.}}$$

- Let $d = \lceil \frac{D}{2} \rceil$, $b_s^\mathsf{T} = [b_s^0, b_s^1, \ldots, b_s^d]$ and $\overline{x}^\mathsf{T} = [1, x, \ldots, x^d]$, the polynomial reads :

$$p(x) = \sum_s q_s^2(x) = \sum_s \left[\sum_{i=0}^{d} b_s^i x_j^i\right] = \sum_s \left(b_s^\mathsf{T}\overline{x}\right)^2$$

$$= \sum_s \overline{x}^\mathsf{T} b_s b_s^\mathsf{T} \overline{x} = \overline{x}^\mathsf{T} \left[\sum_s b_s b_s^\mathsf{T}\right] \overline{x} = \overline{x}^\mathsf{T} Q \overline{x}$$

Semi-Definite Programming (SDP) III

$$p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_D x^D$$

$$= \overline{x}^\mathsf{T} Q \overline{x} = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^d \end{pmatrix}^\mathsf{T} \begin{pmatrix} q_{0,0} & q_{0,1} & q_{0,2} & \cdots & q_{0,d} \\ q_{1,0} & q_{1,1} & q_{1,2} & \cdots & q_{1,d} \\ q_{2,0} & q_{2,1} & q_{2,2} & \cdots & q_{2,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{d,0} & q_{d,1} & q_{d,2} & \cdots & q_{d,d} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^d \end{pmatrix}.$$

- $p_0, p_1, \ldots, p_D$ are obtained by summing the off-diagonal entries of Q :

$$\begin{cases} p_0 = q_{0,0}, \\ p_1 = q_{1,0} + q_{0,1}, \\ p_2 = q_{2,0} + q_{1,1} + q_{0,2}, \\ \vdots \\ p_{2d-1} = q_{d,d-1} + q_{d-1,d}, \\ p_{2d} = q_{d,d}, \text{if } D \text{ is odd, we have } p_{2d} = 0. \end{cases}$$

UTA-poly and UTADIS-poly

- Marginal utility functions defined as polynomials of degree D :

$$u_j^*(a_j) = \sum_{i=0}^{D} p_{j,i} \cdot a_j^i.$$

- To ensure monotonicity of the function, $u_j^{*'}(a_j)$ has to be nonnegative.
  $\Rightarrow u_j^{*'}$ should be a SOS :

$$\frac{\partial u_j^*}{\partial a_j} = p_{j,1} + 2p_{j,2} \cdot a_j + 3p_{j,3} \cdot a_j^2 + \ldots + Dp_{j,n} \cdot a_j^{D-1}$$
$$= \frac{\partial}{\partial a_j} \left[\overline{a_j}^{\mathsf{T}} Q \overline{a_j} \right]$$

- Using a SDP solver, we impose Q to be semidefinite positive.

UTA-poly - Example I

	x	y
a^1	10	7
a^2	6	8
a^3	7	5

$$a^1 \succ a^2 \succ a^3$$

- We define $u_1^*(x)$ and $u_2^*(y)$ as second degree polynomials :

$$u_1^*(x) = p_{x,0} + p_{x,1} \cdot x + p_{x,2} \cdot x^2,$$
$$u_2^*(y) = p_{y,0} + p_{y,1} \cdot y + p_{y,2} \cdot y^2.$$

- Utilities of a^1, a^2 and a^3 are given by :

$$U(a^1) = p_{x,0} + 10p_{x,1} + 100p_{x,2} + p_{y,0} + 7p_{y,1} + 49p_{y,2},$$
$$U(a^2) = p_{x,0} + 6p_{x,1} + 36p_{x,2} + p_{y,0} + 8p_{y,1} + 64p_{y,2},$$
$$U(a^3) = p_{x,0} + 7p_{x,1} + 49p_{x,2} + p_{y,0} + 5p_{y,1} + 25p_{y,2}.$$

UTA-poly - Example II

- We have $a^1 \succ a^2$ and $a^2 \succ a^3$, which implies :

$$\begin{cases} U(a^1) - U(a^2) + \sigma^+(a^1) - \sigma^-(a^1) - \sigma^+(a^2) + \sigma^-(a^2) & > & 0, \\ U(a^2) - U(a^3) + \sigma^+(a^2) - \sigma^-(a^2) - \sigma^+(a^1) + \sigma^-(a^1) & > & 0. \end{cases}$$

- By replacing $U(a^1)$, $U(a^2)$ and $U(a^3)$, we have :

$$\begin{cases} 4p_{x,1} + 64p_{x,2} - p_{y,1} - 15p_{y,2} + \sigma^+(a^1) - \sigma^-(a^1) & & \\ \qquad -\sigma^+(a^2) + \sigma^-(a^2) & > & 0, \\ -p_{x,1} - 13p_{x,2} + 3p_{y,1} + 39p_{y,2} + \sigma^+(a^2) - \sigma^-(a^2) & & \\ \qquad -\sigma^+(a^3) + \sigma^-(a^3) & > & 0. \end{cases}$$

UTA-poly - Example III

- We impose the derivate of u_1^* and u_2^* to be SOS :

$$\begin{aligned}
\frac{\partial u_1^*}{\partial x} &= \overline{x}^\mathsf{T} Q \overline{x} \\
&= \begin{pmatrix} 1 \\ x \end{pmatrix}^\mathsf{T} \begin{pmatrix} q_{x,0,0} & q_{x,0,1} \\ q_{x,1,0} & q_{x,1,1} \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \\
&= q_{0,0} + (q_{0,1} + q_{0,1})\, x + q_{1,1} x^2, \\
\frac{\partial u_2^*}{\partial y} &= \overline{y}^\mathsf{T} R \overline{y} \\
&= r_{0,0} + (r_{0,1} + r_{1,0})\, y + r_{1,1} y^2.
\end{aligned}$$

- Q and R have to be semi-definite positive, in conjunction with :

$$\begin{cases} p_{x,1} &= q_{0,1} + q_{1,0}, \\ 2p_{x,2} &= q_{1,1}, \end{cases} \quad \text{and} \quad \begin{cases} p_{y,1} &= r_{0,1} + r_{1,0}, \\ 2p_{y,2} &= r_{1,1}. \end{cases}$$

UTA-poly - Example IV

- We add normalization constraints :

$$\begin{cases} p_{x,0} & = & 0, \\ p_{y,0} & = & 0, \\ 10p_{x,1} + 100p_{x,2} + 10p_{y,1} + 100p_{y,2} & = & 1. \end{cases}$$

UTA-poly - Example V

- Finally, we obtain the following program :

$$\min \sigma^+(a^1) + \sigma^-(a^1) + \sigma^+(a^2) + \sigma^-(a^2) + \sigma^+(a^3) + \sigma^-(a^3).$$

such that :

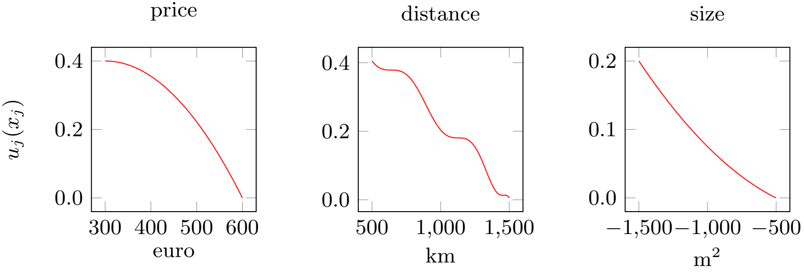
$$\left\{ \begin{array}{rcl} 4p_{x,1} + 64p_{x,2} - p_{y,1} - 15p_{y,2} + \sigma^+(a^1) - \sigma^-(a^1) & & \\ -\sigma^+(a^2) + \sigma^-(a^2) & > & 0, \\ -p_{x,1} - 13p_{x,2} + 3p_{y,1} + 39p_{y,2} + \sigma^+(a^2) - \sigma^-(a^2) & & \\ -\sigma^+(a^3) + \sigma^-(a^3) & > & 0, \\ p_{x,0} & = & 0, \\ p_{y,0} & = & 0, \\ 10p_{x,1} + 100p_{x,2} + 10p_{y,1} + 100p_{y,2} & = & 1, \\ p_{x,1} & = & q_{0,1} + q_{1,0}, \\ 2p_{x,2} & = & q_{1,1}, \\ p_{y,1} & = & r_{0,1} + r_{1,0}, \\ 2p_{y,2} & = & r_{1,1}, \end{array} \right.$$

with :

$$\left\{ \begin{array}{rcl} Q, R & \geq & 0, \\ \sigma^+(a^1), \sigma^-(a^1), \sigma^+(a^2), \sigma^-(a^2), \sigma^+(a^3), \sigma^-(a^3), & \geq & 0. \end{array} \right.$$

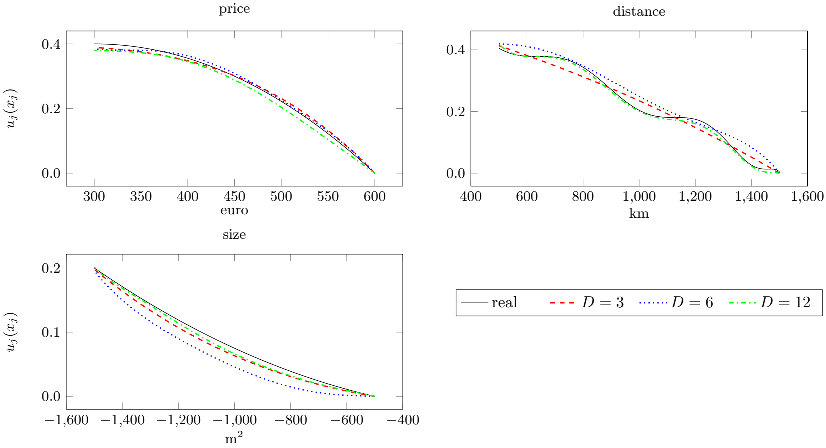
Illustrative example (UTADIS-poly) I

- Evaluation of accommodations for holidays on 3 criteria : price, distance and size.
- 10 categories (from good to bad)
- Consider the following true marginal utility functions

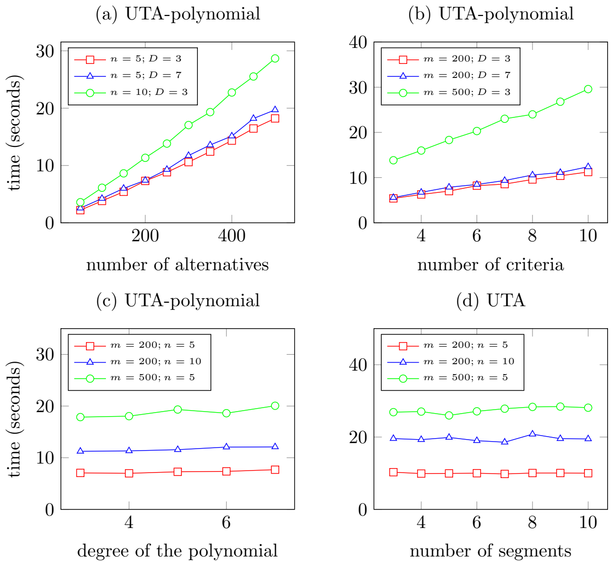


- 200 examples given as input to UTADIS-poly

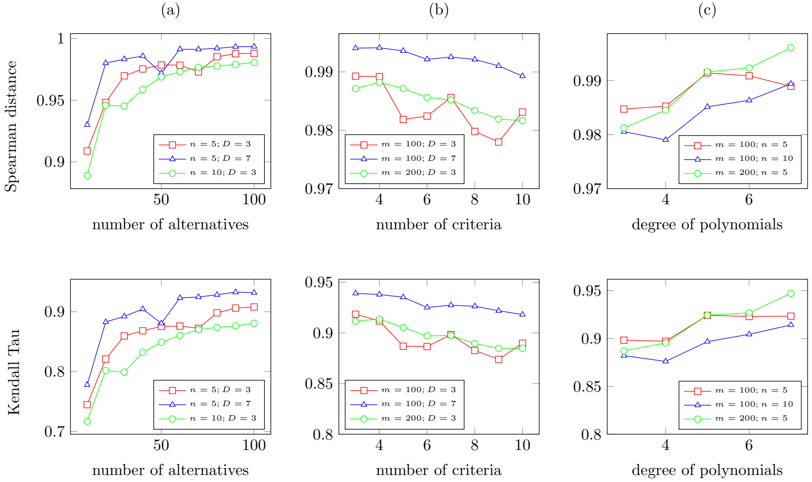
Illustrative example (UTADIS-poly) II



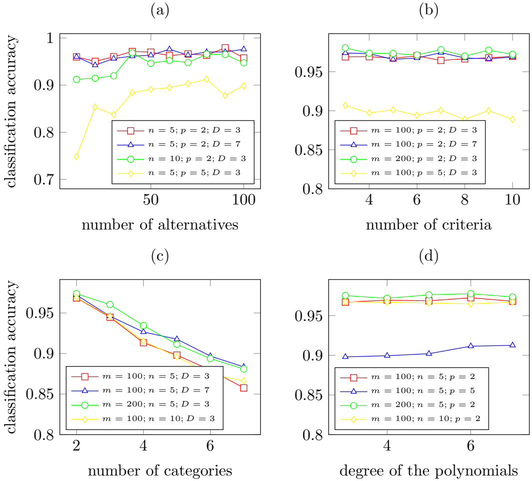
Computing time



Learning a UTA-poly model from a ranking obtained with UTA



Learning a UTADIS-poly model from assignment examples obtained with UTADIS



Conclusion

- More natural marginal utility functions
- Do not increase computing time
- Some marginal utility functions are difficult to model (step)

Thank you for your attention !

References I

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