# Ranking with Multiple reference Points

## Efficient Elicitation and Learning Procedure

Khaled Belahcène[1] - Vincent Mousseau[1] - Wassila Ouerdane[1] - Marc Pirlot[2] - Olivier Sobrie[2]

[1]CentraleSupélec - Université Paris-Saclay
[2]University of Mons - Faculty of engineering

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# Ranking alternatives/objects

## Problem

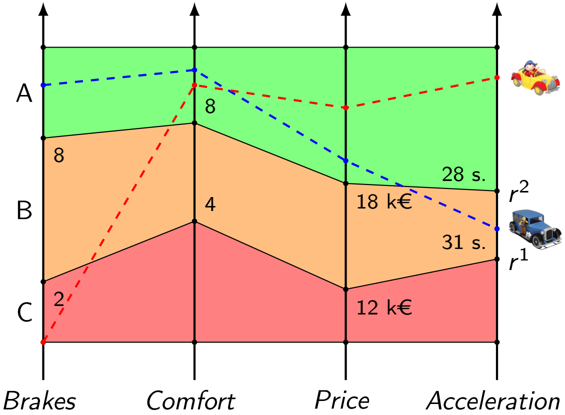
- Ranking alternative/object by preference
- e.g. ranking of cars

 $\succ$  $\succ$ 

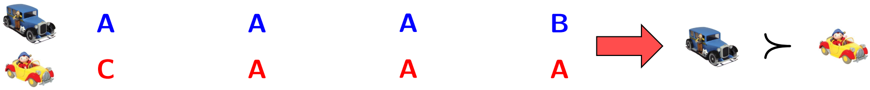
## MCDA ranking methods/models

- UTilités Additives (UTA)
- ELimination and Choice Expressing REality (ELECTRE II)
- Ranking with Multiple reference Points (RMP)

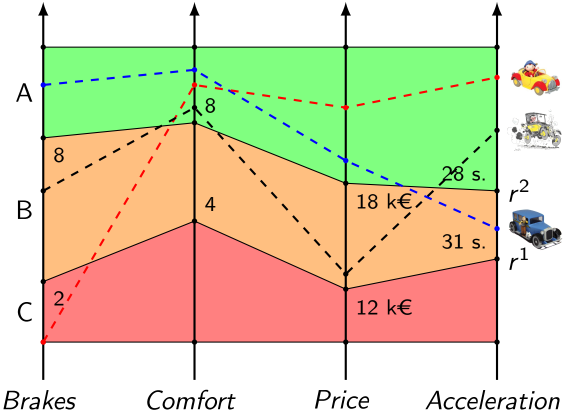
# Ranking with Multiple reference Points (RMP) I




- Equi-important criteria
- Reference points $\mathcal{R} = \{r^1, r^2\}$

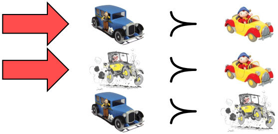


# Ranking with Multiple reference Points (RMP) II




- Equi-important criteria
- Reference points $\mathcal{R} = \{r^1, r^2\}$

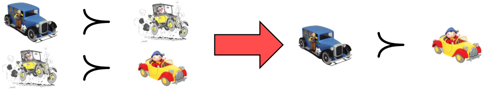
| | | | | |
|---|---|---|---|---|
|  | **A** | **A** | **A** | **B** |
|  | **C** | **A** | **A** | **A** |
|  | **B** | **A** | **B** | **A** |



# Ranking with Multiple reference Points (RMP) III

## Some characteristics of RMP

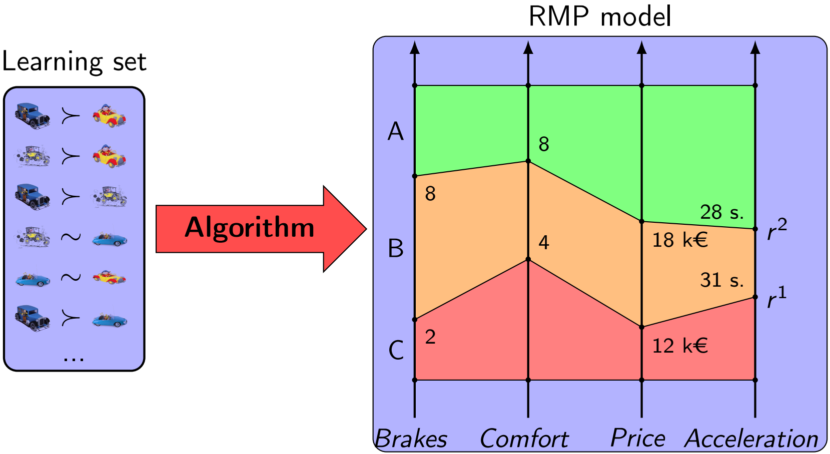
- Model introduced by Antoine Rolland (Rolland, 2013)
- Transitivity ensured



- Safe regarding rank-reversal

- No need for commensurate scales

# Inferring the parameters of an RMP model I



# Inferring the parameters of an RMP model II

## Existing algorithms

- MIP-based algorithms (Zheng et al., 2012; Liu, 2016)
  - S-RMP model (RMP with additive weights)
  - Mixed Integer Program
  - Minimization of Kemeny distance (Kemeny, 1959)
- Metaheuristic algorithm (Liu et al., 2014; Liu, 2016)
  - S-RMP model (RMP with additive weights)
  - Evolutionnary algorithm
  - Reasonable computing time

## Limitations of the existing algorithms

- Additive representation of criteria importance relation
- MIP only able to handle very limited datasets
- Metaheuristic cannot always restore a S-RMP model

# SAT formulation for inferring an RMP Model I

## Boolean Satisfiability problem

- Boolean variables $V$ ;
- Logical proposition about these variables $f : \{0,1\}^V \rightarrow \{0,1\}$ ;
- SATisfiable if $v^*$ exists such that $f(v^*) = 1$
- $f$ can be expressed as conjunction of *clauses* $\mathcal{C}$ :
  $f = \bigwedge_{c \in \mathcal{C}} c$ ;
- Each *clause* $c \in \mathcal{C}$ is a disjunction of their variables or their negation :
  $\forall c \in \mathcal{C}, \exists c^+, c^- \in \mathcal{P}(V) : c = \bigvee_{v \in c^+} v \vee \bigvee_{v \in c^-} \neg v$ ;
- NP-complete problem BUT efficient SAT algorithms exist

## SAT for learning an RMP model

- Expression of constraints as a SAT problem
- Limited to strict preferences ($a \succ b$)

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 1. $\varphi_{\text{scales}}$ : Monotonicity of criteria scales

$$\varphi_{\text{scales}} := \bigwedge_{i \in N} \bigwedge_{k' < k \in \mathbb{X}_i} (x_{i,h,k} \vee \neg x_{i,h,k'})$$

- $x_{i,h,k}$ : equal to 1 if value $k$ above reference point $r^h$ on criterion $i$
- $N$ : set of criteria indices
- $\mathbb{X}_i$ : set of values on criterion $i$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 2. $\varphi_{\text{profiles}}$ : Dominance of the profiles

$$\varphi_{\text{profiles}} := \varphi_{\text{profiles}_1} \wedge \varphi_{\text{profiles}_2}$$

$$\varphi_{\text{profiles}_1} := \bigwedge_{h \neq h' \in H} \bigwedge_{i \in N} \bigwedge_{k \in \mathbb{X}_i} (x_{i,h',k} \vee \neg x_{i,h,k} \vee \neg d_{h,h'})$$

$$\varphi_{\text{profiles}_2} := \bigwedge_{h < h' \in H} (d_{h,h'} \vee d_{h',h})$$

- $N$ : set of criteria indices
- $\mathbb{X}_i$ : set of values on criterion $i$
- $H$ : set of reference points indices
- $d_{h,h'}$ : equal to 1 if value if $r^h$ dominates $r^{h'}$
- $x_{i,h,k}$ : equal to 1 if value $k$ above reference point $r^h$ on criterion $i$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 3. $\varphi_{\text{order}}$ : Order among criteria sets

$$\varphi_{\text{order}} := \varphi_{\text{Pareto}} \wedge \varphi_{\text{completeness}} \wedge \varphi_{\text{transitivity}}$$

$$\varphi_{\text{Pareto}} := \bigwedge_{A \subseteq B \in \mathcal{P}(N)} (y_{B,A})$$

$$\varphi_{\text{completeness}} := \bigwedge_{A,B \in \mathcal{P}(N)} (y_{A,B} \vee y_{B,A})$$

$$\varphi_{\text{transitivity}} := \bigwedge_{A,B,C \in \mathcal{P}(N)} (\neg y_{A,B} \vee \neg y_{B,C} \vee y_{A,C})$$

- $\mathcal{P}(N)$ : set of possible criteria coalitions
- $y_{A,B}$ : equal to 1 if criteria coalition $A$ is more important than criteria coalition $B$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 4. $\varphi_{\text{outranking}}$ : Outranking relation between pairs

$$\varphi_{\text{outranking}} := \varphi_{\text{outranking}_1} \wedge \varphi_{\text{outranking}_2} \wedge \varphi_{\text{outranking}_3}$$

$$\varphi_{\text{outranking}_1} := \bigwedge_{A,B \in \mathcal{P}(N)} \bigwedge_{j \in J} \bigwedge_{h \in H} \left( \bigvee_{i \notin A} x_{i,h,p_i^j} \vee \bigvee_{i \in B} \neg x_{i,h,n_i^j} \vee y_{A,B} \vee \neg z_{j,h} \right)$$

- $p_j \succ n_j$ : pairwise comparison $j$
- $J$ : set of pairwise comparisons indices
- $\mathcal{P}(N)$ : set of possible criteria coalitions
- $H$ : set of reference points indices
- $x_{i,h,k}$ : equal to 1 if value $k$ above reference point $r^h$ on criterion $i$
- $y_{A,B}$ : equal to 1 if criteria coalition $A$ is more important than criteria coalition $B$
- $z_j$ : equals to 1 if criteria set on which $p_j$ above $r^h$ is more important than the criteria set on which $n_j$ is above $r^h$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 4. $\varphi_{\text{outranking}}$ : Outranking relation between pairs

$$\varphi_{\text{outranking}} := \varphi_{\text{outranking}_1} \wedge \varphi_{\text{outranking}_2} \wedge \varphi_{\text{outranking}_3}$$

$$\varphi_{\text{outranking}_2} := \bigwedge_{A,B \in \mathcal{P}(N)} \bigwedge_{j \in J} \bigwedge_{h \in H} \Big( \bigvee_{i \notin A} x_{i,h,n_i^j} \vee \bigvee_{i \in B} \neg x_{i,h,p_i^j} \vee y_{A,B} \vee \neg z'_{j,h} \Big)$$

- $p_j \succ n_j$ : pairwise comparison $j$
- $J$ : set of pairwise comparisons indices
- $\mathcal{P}(N)$ : set of possible criteria coalitions
- $H$ : set of reference points indices
- $x_{i,h,k}$ : equal to 1 if value $k$ above reference point $r^h$ on criterion $i$
- $y_{A,B}$ : equal to 1 if criteria coalition $A$ is more important than criteria coalition $B$
- $z'_j$ : equals to 1 if criteria set on which $n_j$ above $r^h$ is more important than the criteria set on which $p_j$ is above $r^h$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 4. $\varphi_{\text{outranking}}$ : Outranking relation between pairs

$$\varphi_{\text{outranking}} := \varphi_{\text{outranking}_1} \wedge \varphi_{\text{outranking}_2} \wedge \varphi_{\text{outranking}_3}$$

$$\varphi_{\text{outranking}_3} := \bigwedge_{A,B \in \mathcal{P}(N)} \bigwedge_{j \in J} \bigwedge_{h \in H} \Big( \bigvee_{i \in A} \neg x_{i,h,p_i^j} \vee \bigvee_{i \notin A} x_{i,h,p_i^j} \vee \bigvee_{i \in B} \neg x_{i,h,n_i^j} \vee \bigvee_{i \notin B} x_{i,h,n_i^j} \vee \neg y_{B,A} \vee z'_{j,h} \Big)$$

- $p_j \succ n_j$ : pairwise comparison $j$
- $J$ : set of pairwise comparisons indices
- $\mathcal{P}(N)$ : set of possible criteria coalitions
- $H$ : set of reference points indices
- $x_{i,h,k}$ : equal to 1 if value $k$ above reference point $r^h$ on criterion $i$
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# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 5. $\varphi_{\text{lexicography}}$ : Lexicography of reference points

$$\varphi_{\text{lexicography}} := \varphi_{\text{lexicography}_1} \wedge \varphi_{\text{lexicography}_2} \wedge \varphi_{\text{lexicography}_3}$$

$$\varphi_{\text{lexicography}_1} := \bigwedge_{j \in J} \quad \bigwedge_{h \leq h' \in H} (z_{j,h} \vee \neg s_{j,h'})$$

$$\varphi_{\text{lexicography}_2} := \bigwedge_{j \in J} \quad \bigwedge_{h < h' \in H} (z'_{j,h} \vee \neg s_{j,h'})$$

$$\varphi_{\text{lexicography}_3} := \bigwedge_{h \in H} (\neg z'_{j,h} \vee \neg s_{j,h})$$

- $J$ : set of pairwise comparisons indices
- $H$ : set of reference points indices
- $z_j$ : equals to 1 if criteria set on which $p_j$ above $r^h$ is more important than the criteria set on which $n_j$ is above $r^h$
- $z'_j$ : equals to 1 if criteria set on which $n_j$ above $r^h$ is more important than the criteria set on which $p_j$ is above $r^h$
- $s_{j,h}$ : equals to 1 if $p^j$ indifferent to $n^j$ for all reference points $r^{h'}$ such that $h' < h$ and strictly outranks $n^j$ at reference point $r^h$

# SAT formulation for inferring an RMP Model II

$$\varphi := \varphi_{\text{scales}} \wedge \varphi_{\text{profiles}} \wedge \varphi_{\text{order}} \wedge \varphi_{\text{outranking}} \wedge \varphi_{\text{lexicography}} \wedge \varphi_{\text{preference}}$$

## 6. $\varphi_{\text{preference}}$ : Strict preference

$$\varphi_{\text{preference}} := \bigwedge_{j \in J} \Big( \bigvee_{h \in H} s_{j,h} \Big)$$

- $J$ : set of pairwise comparisons indices
- $H$ : set of reference points indices
- $s_{j,h}$ : equals to 1 if $p^j$ indifferent to $n^j$ for all reference points $r^{h'}$ such that $h' < h$ and strictly outranks $n^j$ at reference point $r^h$

# MAX-SAT formulation

## Goal

- Handle incompatibilities in the learning set (e.g. $a \succ b$ and $b \succ a$)

## How ?

- By allowing the violation of some clauses
- Weighting the clauses :
  - Some cannot be violated (e.g. monotonicity of the scales) $\Rightarrow$ Huge weights for these clauses
  - Others can be violated (e.g. pairwise comparison) $\Rightarrow$ Small weights for these clauses

## MAX-SAT solvers

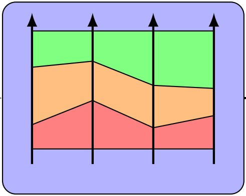
- Lot of different ones (MAX-SAT competition every year)
- Chosen solvers : MaxHS (Davies, 2013), maxino (Alviano et al., 2015).

# Experimental results I

Pairs of alternatives

(3, 5, 2, 7) ? (2, 7, 3, 9)
(1, 2, 3, 4) ? (4, 1, 2, 1)
(4, 3, 6, 5) ? (2, 7, 5, 9)
(5, 9, 7, 6) ? (2, 8, 3, 1)
(2, 6, 4, 7) ? (2, 7, 1, 4)
...

$\mathcal{M}^0$



Learning set

(3, 5, 2, 7) $\succ$ (2, 7, 3, 9)
(1, 2, 3, 4) $\prec$ (4, 1, 2, 1)
(4, 3, 6, 5) $\succ$ (2, 7, 5, 9)
(5, 9, 7, 6) $\prec$ (2, 8, 3, 1)
(2, 6, 4, 7) $\succ$ (2, 7, 1, 4)
...

Learning set

(3, 5, 2, 7) $\succ$ (2, 7, 3, 9)
(1, 2, 3, 4) $\succ$ (4, 1, 2, 1)
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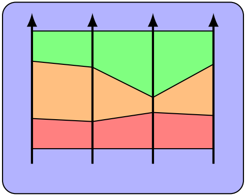
(5, 9, 7, 6) $\prec$ (2, 8, 3, 1)
(2, 6, 4, 7) $\succ$ (2, 7, 1, 4)
...

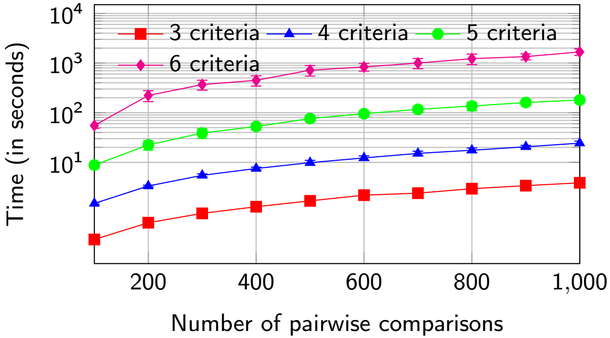
SAT-RMP

$\mathcal{M}^{\text{learned}}$



# Experimental results II

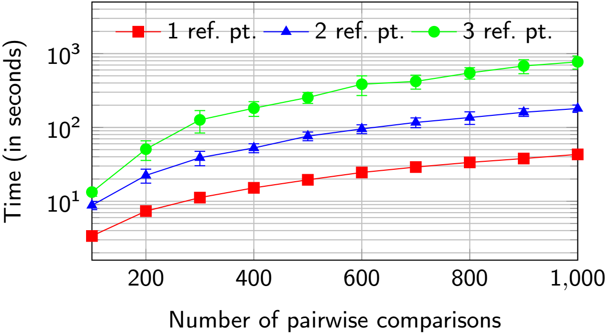
## Computing time




- 2 reference points
- Computing time grows fast when the number of criteria increases
- More efficient than MIP based algorithms

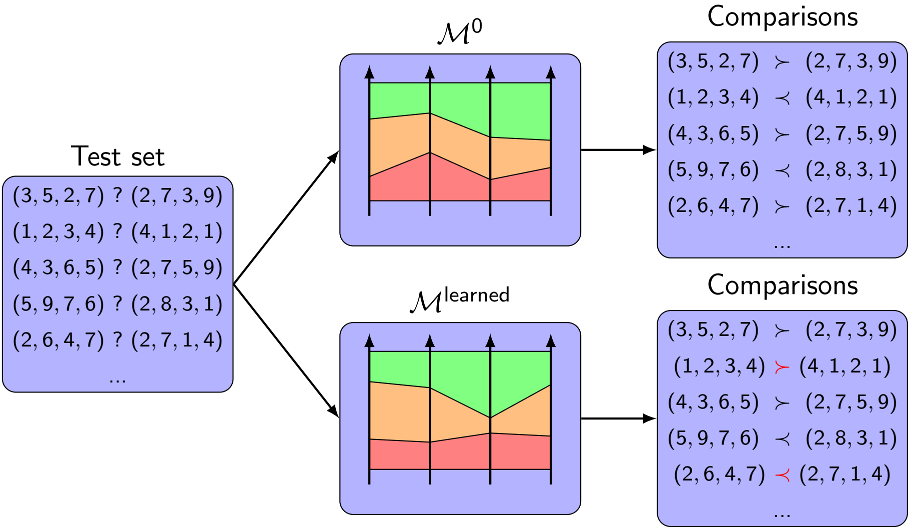
# Experimental results III

## Computing time



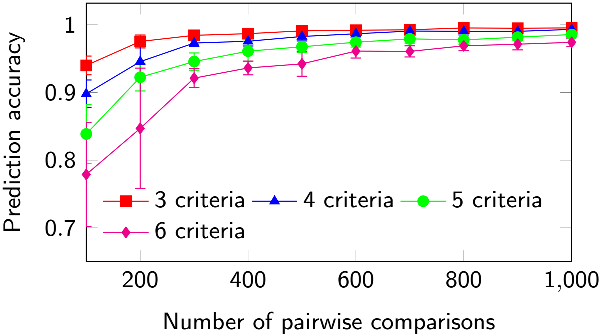

- 5 criteria
- Computing time also significantly impacted by the number of reference points

# Experimental results IV



# Experimental results V

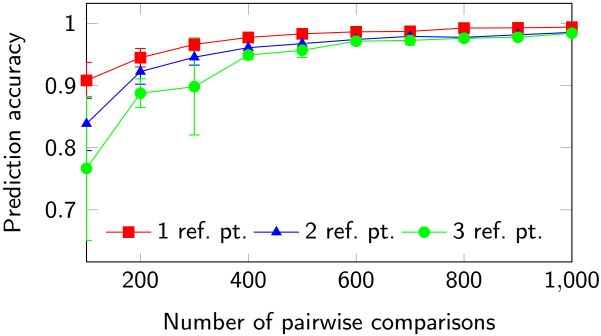
## Model retrieval




- 2 reference points
- Accuracy above 90% with barely 300 pairwise comparisons

# Experimental results VI

## Model retrieval




- 5 criteria
- Number of reference points hasn't lot of impact on the accuracy when the number of pairwise comparisons is greater than 300

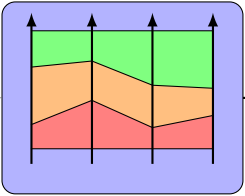
# Experimental results VII

**Pairs of alternatives**

(3, 5, 2, 7) ? (2, 7, 3, 9)
(1, 2, 3, 4) ? (4, 1, 2, 1)
(4, 3, 6, 5) ? (2, 7, 5, 9)
(5, 9, 7, 6) ? (2, 8, 3, 1)
(2, 6, 4, 7) ? (2, 7, 1, 4)
...

→

$\mathcal{M}^0$



→

**Learning set**

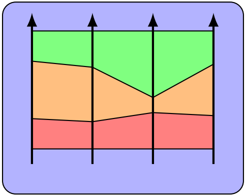
$(3, 5, 2, 7) \succ (2, 7, 3, 9)$
$(1, 2, 3, 4) \prec (4, 1, 2, 1)$
$(4, 3, 6, 5) \succ (2, 7, 5, 9)$
$(5, 9, 7, 6) \prec (2, 8, 3, 1)$
$(2, 6, 4, 7) \succ (2, 7, 1, 4)$
...

**Learning set***

$(3, 5, 2, 7) \succ (2, 7, 3, 9)$
$(1, 2, 3, 4) \succ (4, 1, 2, 1)$
$(4, 3, 6, 5) \succ (2, 7, 5, 9)$
$(5, 9, 7, 6) \prec (2, 8, 3, 1)$
$(2, 6, 4, 7) \prec (2, 7, 1, 4)$
...

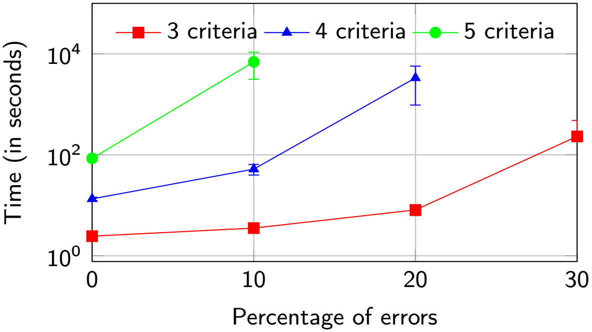
**MAXSAT-RMP** →

$\mathcal{M}^{\text{learned}}$



# Experimental results VIII

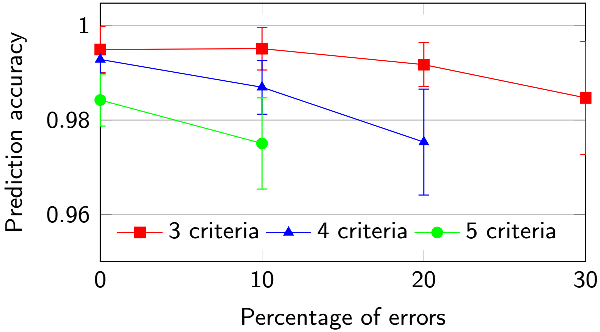
## Tolerance for errors




- 1 profile
- 500 pairwise comparisons
- Computing time increases quickly when the number of inversion increases

# Experimental results IX

## Tolerance for errors




- 1 profiles
- 500 pairwise comparisons
- MAX-SAT formulation identifies errors

# Conclusion and further research

## Conclusion

- Efficient formulation for problems involving less than 7 criteria (1000 pairwise comparisons)
- Computing time increases when there are errors in the learning set

## Further research

- Support for indifference
- Formalization of SAT clauses (demonstration)


**İlginiz için teşekkürler !**

**(Thank you for your attention !)**

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